Vietnam Journal of Mechanics, Vol. 47, No. 2 (2025), pp. 109 – 120
DOI: https://doi.org/10.15625/0866-7136/21845

WEAKLY NONLOCAL RAYLEIGH WAVES IN ORTHOTROPIC HALF-SPACES COATED BY AN ORTHOTROPIC LAYER WITH SLIDING CONTACT

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Received: 29 October 2024 / Revised: 08 March 2025 / Accepted: 16 March 2025
Published online: 21 March 2025

Abstract. In this paper, we study the propagation of Rayleigh waves in nonlocal orthotropic half-spaces coated by a nonlocal orthotropic layer with sliding contact using the weakly nonlocal elasticity model. This model was recently introduced and different from other existing nonlocal models it has been proven to be well-posed for all harmonic plane wave problems. The transfer matrix method and the effective boundary condition method are employed for deriving the explicit dispersion equation of Rayleigh waves. Using the obtained dispersion equation, the effect of the nonlocality and the thickness of the layer on the velocity of Rayleigh waves is considered through some numerical examples. It is shown that the nonlocality and the thickness of the layer strongly affect the velocity of Rayleigh waves and they make it decreasing. Since the dispersion equation of Rayleigh waves is totally explicit, it will be a powerful tool for monitoring the health of the layer/half-space structures during loading.

Keywords: Rayleigh waves, weakly nonlocal elasticity model, sliding contact, explicit dispersion equation.

1. INTRODUCTION

The structure of an elastic half-space coated by an elastic layer has a wide range of applications such as those in seismology, acoustics, geophysics, materials science and micro-electro-mechanical systems. The nondestructive measurement of mechanical properties of these structures before and during loading is therefore important and significant. As we know, Rayleigh waves are a convenient tool for this task [1]. When using Rayleigh waves, their explicit secular equations are employed as mathematical basis for extracting the mechanical properties of layers from experimental data (measured values of wave velocity). Thus, they are the main purpose of any investigation on Rayleigh waves propagating in elastic half-spaces covered by an elastic layer.

With the rapid development of science and technology, devices in different applications such as medical devices, electronic devices, computer chips, etc. are required to be smaller and smaller. Thus, nano-scale materials and structures have received intensive attention of researchers. The classical elasticity is unable to predict properly the nature of such nano-scale materials. For instance, the behaviour of materials with fractures, dislocation, cracks, singularities and discontinuities can not be treated completely by the classical (local) elasticity theory [2–4]. Therefore, to explain and predict the physical phenomena at small length scales, that is, at nano-scales, a series of nonlocal continuum mechanics have been developed including Eringen's fully nonlocal model [5], Eringen's two-phase local/nonlocal model [6, 7], stress-driven nonlocal model [8], nonlocal strain gradient model [9] and weakly nonlocal elasticity model [10]. Among these nonlocal elasticity models, the weakly nonlocal elasticity model has been proven to be well-posed for any harmonic plane wave problem [10, 11], Eringen's fully nonlocal model has been shown to be ill-posed for harmonic plane wave problems of which constitutive boundary conditions do not contain all equilibrium boundary conditions [12], and it is still unclear whether the remaining models are well-posed or not for problems of harmonic plane waves. Therefore, the weakly nonlocal elasticity model proposed recently by Anh and Vinh [10] is the best choice as a theoretical model for studying the propagation of nonlocal harmonic plane waves.

In this paper, we employ the weakly nonlocal elasticity theory to study the propagation of Rayleigh waves in nonlocal orthotropic half-spaces coated by a nonlocal orthotropic layer with sliding contact. The main aim is to derive the explicit dispersion equation of nonlocal Rayleigh waves. It has been obtained by employing the transfer matrix method along with the effective boundary condition technique. Based on it, the effect of the nonlocality and the thickness of the layer on the velocity of Rayleigh waves is considered through some numerical examples. It is shown numerically that the nonlocality (of the layer and the half-space) and the thickness of layer strongly affect the velocity of Rayleigh waves and they make it decreasing. Since the dispersion equation of Rayleigh waves is totally explicit, it will be a powerful tool for monitoring the health of the layer/half-space structures during loading. It should also be noted that the weakly nonlocal elasticity theory has been employed to study the propagation of nonlocal Rayleigh waves [11, 13], nonlocal Stoneley waves [10] and nonlocal Lamb waves in a layered nonlocal elastic layer [14].

There have been several investigations on Rayleigh waves propagating in nonlocal elastic half-spaces coated by a nonlocal elastic layer. They are the investigation [15] for Rayleigh waves in a nonlocal thermoelastic layer lying over a nonlocal thermoelastic half-space, the work [16] for Rayleigh waves in a porous nonlocal orthotropic thermoelastic layer lying over porous nonlocal orthotropic thermoelastic half-space, the studies [17,18] for Rayleigh waves in a layered structure constituted of nonlocal functionally gradient transversely isotropic stratum lying over a nonlocal functionally gradient monoclinic substrate and the contribution [13] for Rayleigh waves in a nonlocal orthotropic elastic half-space coated by a nonlocal orthotropic elastic layer. It is noted that the Rayleigh wave problems studied in investigations [15–18] have no solutions, according to the well-posedness criterion for harmonic plane wave problems of Eringen's nonlocal elasticity theory [12]. These problems will be reconsidered by the weakly nonlocal elasticity theory [10,11] which has been proven to be well-posed for any harmonic plane wave problem.

2. TRANSFER MATRIX FOR A WEAKLY NONLOCAL ORTHOTROPIC LAYER

Consider a nonlocal orthotropic elastic layer of uniform thickness h occupying the domain $a \leq x_2 \leq b, b - a = h$. We are interested in the in-plane motion in the plane (x_1, x_2) whose displacement components are of the form

$$u_i = u_i\left(x_1, x_2, t\right), \quad i = 1, 2, \quad u_3 \equiv 0, \tag{1}$$

where t is the time. Suppose the layer's material is weakly nonlocal elastic (see Anh and Vinh [10], Anh et al. [11]). Then, in the absence of body forces, the equations of motion are

$$t_{11,1} + t_{12,2} = \rho\ddot{u}_1, \quad t_{12,1} + t_{22,2} = \rho\ddot{u}_2, \tag{2}$$

where ρ is the mass density, subscript commas signify differentiation with respect to x_k and a dot represents the partial time derivative with respect to t; t_{ij} are the nonlocal stress components that are related to the corresponding local stresses σ_{ij} by the differential relations

$$\sigma_{11} = (1 - \epsilon^2\nabla^2)t_{11}, \quad \sigma_{12} = (1 - \epsilon^2\nabla^2)t_{12}, \quad \sigma_{22} = (1 - \epsilon^2\nabla^2)t_{22}, \tag{3}$$

along with the extra conditions

$$\sigma_{ij} \equiv 0 \Rightarrow t_{ij} \equiv 0 \quad (i, j = 1, 2), \tag{4}$$

where $\nabla = \left[\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right]^T$ and $\epsilon = e_0 l$ is the nonlocal parameter, with e_0 is the material constant and l is the internal characteristic length. Applying operation $L = 1 - \epsilon^2\nabla^2$ to the original equations of motion (2) and using (3) yield

$$\sigma_{11,1} + \sigma_{12,2} = \rho\ddot{u}_1 - \rho\epsilon^2(\ddot{u}_{1,11} + \ddot{u}_{1,22}), \quad \sigma_{12,1} + \sigma_{22,2} = \rho\ddot{u}_2 - \rho\epsilon^2(\ddot{u}_{2,11} + \ddot{u}_{2,22}). \tag{5}$$

Since the layer is made of compressible orthotropic elastic materials, the local stresses σ_{ij} are expressed in terms of displacements by

$$\sigma_{11} = c_{11}u_{1,1} + c_{12}u_{2,2}, \quad \sigma_{22} = c_{12}u_{1,1} + c_{22}u_{2,2}, \quad \sigma_{12} = c_{66}\left(u_{1,2} + u_{2,1}\right), \tag{6}$$

where $c_{ij}\ (i,j = 1,2,6)$ are the stiffness elastic constants. Using (6) into (5), we obtain

$$\begin{aligned} c_{11}u_{1,11} + c_{66}u_{1,22} + (c_{12} + c_{66})u_{2,12} &= \rho\ddot{u}_1 - \rho\epsilon^2(\ddot{u}_{1,11} + \ddot{u}_{1,22}), \\ (c_{12} + c_{66})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} &= \rho\ddot{u}_2 - \rho\epsilon^2(\ddot{u}_{2,11} + \ddot{u}_{2,22}). \end{aligned} \tag{7}$$

Since the operator L is invertible due to the extra conditions (4), the original equations of motion (2) are equivalent to the resulting equations of motion (7). Hence, Eqs. (7) will be used instead of equations (2). Consider a plane wave propagating in the x_1-direction with velocity $c\ (> 0)$ and wavenumber $k\ (> 0)$. Then, its displacement and stress components are of the form

$$\begin{aligned} u_1 &= U_1(x_2)\mathrm{e}^{ik(x_1 - ct)}, \quad u_2 = U_2(x_2)\mathrm{e}^{ik(x_1 - ct)}, \\ t_{12} &= kT_1(x_2)\mathrm{e}^{ik(x_1 - ct)}, \quad t_{22} = kT_2(x_2)\mathrm{e}^{ik(x_1 - ct)}, \end{aligned} \tag{8}$$

where $U_j(x_2)$ and $T_j(x_2)\ (j = 1,2)$ are functions to be determined. Define $\mathbf{U} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$, $\mathbf{T} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^T$ and $\boldsymbol{\xi}(x_2) = \begin{bmatrix} \mathbf{U}(x_2) & \mathbf{T}(x_2) \end{bmatrix}^T$. According to Anh et al. [14], we have

$$\boldsymbol{\xi}\,(a) = \mathbf{N}\boldsymbol{\xi}\,(b)\,, \tag{9}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}, \tag{10}$$

matrices $\mathbf{N}_j\ (j = 1,2,3,4)$ are given by

$$\mathbf{N}_1 = \begin{bmatrix} \dfrac{[\gamma; \mathrm{ch}\varepsilon]}{[\gamma]} & -\dfrac{i\,[\beta; \mathrm{sh}\varepsilon]}{[\alpha; \beta]} \\[2ex] \dfrac{i\,[\alpha\mathrm{sh}\varepsilon; \gamma]}{[\gamma]} & \dfrac{[\alpha\mathrm{ch}\varepsilon; \beta]}{[\alpha; \beta]} \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} -\dfrac{[\alpha; \mathrm{sh}\varepsilon]}{[\alpha; \beta]} & -\dfrac{i\,[\mathrm{ch}\varepsilon]}{[\gamma]} \\[2ex] -\dfrac{i\alpha_1\alpha_2\,[\mathrm{ch}\varepsilon]}{[\alpha; \beta]} & -\dfrac{[\alpha\mathrm{sh}\varepsilon]}{[\gamma]} \end{bmatrix},$$

$$\mathbf{N}_3 = \begin{bmatrix} \dfrac{[\beta\mathrm{sh}\varepsilon; \gamma]}{[\gamma]} & -\dfrac{i\beta_1\beta_2\,[\mathrm{ch}\varepsilon]}{[\alpha; \beta]} \\[2ex] -\dfrac{i\gamma_1\gamma_2\,[\mathrm{ch}\varepsilon]}{[\gamma]} & \dfrac{[\beta; \gamma\mathrm{sh}\varepsilon]}{[\alpha; \beta]} \end{bmatrix}, \quad \mathbf{N}_4 = \begin{bmatrix} \dfrac{[\alpha; \beta\mathrm{ch}\varepsilon]}{[\alpha; \beta]} & \dfrac{i\,[\beta\mathrm{sh}\varepsilon]}{[\gamma]} \\[2ex] -\dfrac{i\,[\alpha; \gamma\mathrm{sh}\varepsilon]}{[\alpha; \beta]} & \dfrac{[\gamma\mathrm{ch}\varepsilon]}{[\gamma]} \end{bmatrix}. \tag{11}$$

Here, we use the following notations

$$[f] = f_2 - f_1, \quad [f; g] = f_2 g_1 - f_1 g_2, \tag{12}$$

and $\alpha_j, \beta_j, \gamma_j$ in (11) are calculated by

$$\begin{aligned}
&\alpha_j = \frac{(c_{12}+c_{66})b_j}{c_{66}^* - X - c_{22}^* b_j^2}, \quad \beta_j = \frac{c_{66}(b_j - \alpha_j)}{1+e(1-b_j^2)}, \quad \gamma_j = \frac{c_{12}+c_{22}b_j\alpha_j}{1+e(1-b_j^2)}, \quad j=1,2 \\
&b_1 = \sqrt{\frac{S+\sqrt{S^2-4P}}{2}}, \quad b_2 = \sqrt{\frac{S-\sqrt{S^2-4P}}{2}}, \\
&S = -\frac{c_{22}^*(X-c_{11}^*)+c_{66}^*(X-c_{66}^*)+(c_{12}+c_{66})^2}{c_{22}^*c_{66}^*}, \quad P = \frac{(c_{11}^*-X)(c_{66}^*-X)}{c_{22}^*c_{66}^*},
\end{aligned} \tag{13}$$

where

$$c_{22}^* = c_{22} - Xe, \quad c_{66}^* = c_{66} - Xe, \quad c_{11}^* = c_{11} - Xe, \quad X = \rho c^2, \quad e = k^2\epsilon^2, \tag{14}$$

and in (11) we have: $\varepsilon_j = \varepsilon b_j\,(j=1,2), \varepsilon = kh$. From (9) we have

$$\boldsymbol{\xi}(b) = \hat{\mathbf{N}}\boldsymbol{\xi}(a), \quad \hat{\mathbf{N}} = \mathbf{N}^{-1}. \tag{15}$$

By changing the direction of x_2-axis, we can see that matrix $\hat{\mathbf{N}}$ is determined by (11) in which ε is replaced by $-\varepsilon$. In particular, matrix $\hat{\mathbf{N}}$ is of the form

$$\hat{\mathbf{N}} = \begin{bmatrix} \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 \\ \hat{\mathbf{N}}_3 & \hat{\mathbf{N}}_4 \end{bmatrix}, \tag{16}$$

with matrices $\hat{\mathbf{N}}_j$ $(j=\overline{1,4})$ are determined by

$$\hat{\mathbf{N}}_1 = \begin{bmatrix} \dfrac{[\gamma;\mathrm{ch}\varepsilon]}{[\gamma]} & \dfrac{i\,[\beta;\mathrm{sh}\varepsilon]}{[\alpha;\beta]} \\[2ex] -\dfrac{i\,[\alpha\mathrm{sh}\varepsilon;\gamma]}{[\gamma]} & \dfrac{[\alpha\mathrm{ch}\varepsilon;\beta]}{[\alpha;\beta]} \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} \dfrac{[\alpha;\mathrm{sh}\varepsilon]}{[\alpha;\beta]} & -\dfrac{i\,[\mathrm{ch}\varepsilon]}{[\gamma]} \\[2ex] -\dfrac{i\alpha_1\alpha_2\,[\mathrm{ch}\varepsilon]}{[\alpha;\beta]} & \dfrac{[\alpha\mathrm{sh}\varepsilon]}{[\gamma]} \end{bmatrix},$$

$$\hat{\mathbf{N}}_3 = \begin{bmatrix} -\dfrac{[\beta\mathrm{sh}\varepsilon;\gamma]}{[\gamma]} & -\dfrac{i\beta_1\beta_2\,[\mathrm{ch}\varepsilon]}{[\alpha;\beta]} \\[2ex] -\dfrac{i\gamma_1\gamma_2\,[\mathrm{ch}\varepsilon]}{[\gamma]} & -\dfrac{[\beta;\gamma\mathrm{sh}\varepsilon]}{[\alpha;\beta]} \end{bmatrix}, \quad \hat{\mathbf{N}}_4 = \begin{bmatrix} \dfrac{[\alpha;\beta\mathrm{ch}\varepsilon]}{[\alpha;\beta]} & -\dfrac{i[\beta\mathrm{sh}\varepsilon]}{[\gamma]} \\[2ex] \dfrac{i\,[\alpha;\gamma\mathrm{sh}\varepsilon]}{[\alpha;\beta]} & \dfrac{[\gamma\mathrm{ch}\varepsilon]}{[\gamma]} \end{bmatrix}. \tag{17}$$

Matrices $\mathbf{N}$ and $\hat{\mathbf{N}}$ given by (10) and (16) are called the transfer matrices for a weakly nonlocal compressible orthotropic elastic layer. Note that in deriving (10)–(17) we assume that $b_1 \neq b_2$.

3. EXPLICIT DISPERSION EQUATION OF RAYLEIGH WAVES

3.1. Effective boundary condition

Let us consider a medium which consists of a nonlocal orthotropic layer of arbitrary thickness h occupying the domain $-h \leq x_2 \leq 0$ lying over a nonlocal orthotropic half-space $x_2 \geq 0$. It is assumed that the contact between the half-space and the layer is a sliding contact and the top surface of the layer $x_2 = -h$ is free from traction. Note that the same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer.

From the traction-free condition: $\bar{t}_{12} = 0$ and $\bar{t}_{22} = 0$ at $x_2 = -h$, using (9), (10) and (11) with $a = -h, b = 0$, we obtain

$$\begin{aligned} &n_{31}\bar{U}_1(0) + n_{32}\bar{U}_2(0) + n_{33}\bar{T}_1(0) + n_{34}\bar{T}_2(0) = 0, \\ &n_{41}\bar{U}_1(0) + n_{42}\bar{U}_2(0) + n_{43}\bar{T}_1(0) + n_{44}\bar{T}_2(0) = 0, \end{aligned} \tag{18}$$

where n_{31}, n_{32} and n_{41}, n_{42} are respectively elements of the first row and the second row of matrix $\mathbf{N}_3$; n_{33}, n_{34} and n_{43}, n_{44} are respectively elements of the first row and the second row of matrix $\mathbf{N}_4$; $\mathbf{N}_3$ and $\mathbf{N}_4$ are determined by (11). Since the contact between the layer and the half-space is a sliding contact, we have

$$t_{12} = 0, \quad \bar{t}_{12} = 0, \quad t_{22} = \bar{t}_{22}, \quad u_2 = \bar{u}_2 \quad \text{at} \quad x_2 = 0, \tag{19}$$

or, in view of the relations (8)

$$T_1(0) = 0, \quad \bar{T}_1(0) = 0, \quad T_2(0) = \bar{T}_2(0), \quad U_2(0) = \bar{U}_2(0). \tag{20}$$

Using (20) into (18) yields

$$(n_{32}n_{41} - n_{31}n_{42})U_2(0) + (n_{34}n_{41} - n_{31}n_{44})T_2(0) = 0. \tag{21}$$

From the first of (20) and (21), we see that the surface $x_2 = 0$ of the half-space is subjected to the following conditions

$$\begin{aligned} &T_1(0) = 0, \\ &(n_{32}n_{41} - n_{31}n_{42})U_2(0) + (n_{34}n_{41} - n_{31}n_{44})T_2(0) = 0. \end{aligned} \tag{22}$$

The second of (22) is the desired effective boundary condition. The total effect of the layer on the half-space is replaced by this condition.

3.2. Explicit dispersion equation of Rayleigh waves

Now we can ignore the layer and consider the propagation of Rayleigh waves in the nonlocal orthotropic elastic half-space $x_2 \geq 0$ subject to the boundary conditions (22) with wave velocity c (> 0) and wavenumber k (> 0) in the x_1-direction and decaying in the x_2-direction. It is not difficult to verify that the displacements of the Rayleigh wave in the half-space are given by $(8)_{1,2}$ with $U_1(y)$ and $U_2(y)$ are determined by

$$U_1(y) = B_1 e^{-b_1 y} + B_2 e^{-b_2 y}, \quad U_2(y) = i\Big(\alpha_1 B_1 e^{-b_1 y} + \alpha_2 B_2 e^{-b_2 y}\Big), \tag{23}$$

with B_1 and B_2 are constants to be determined and

$$\alpha_j = -\frac{(c_{12}+c_{66})b_j}{c_{66}^* - X - c_{22}^* b_j^2}, \quad j = 1,2 \tag{24}$$

in which $c_{66}^* = c_{66} - Xe$, $c_{22}^* = c_{22} - Xe$, $X = \rho c^2$, $e = \epsilon^2 k^2$, b_1 and b_2 are two roots with positive real part of the following characteristic equation

$$b^4 - Sb^2 + P = 0, \tag{25}$$

where S, P are given by $(13)_{6,7}$. One can see that if a Rayleigh wave exists ($\Rightarrow$ the real parts of b_1 and b_2 must be positive) it implies (see also Vinh [19] and Vinh & Seriani [20])

$$P > 0, \quad S + 2\sqrt{P} > 0, \quad b_1 b_2 = \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}. \tag{26}$$

From (23), $(8)_{1,2}$ and (6) we deduce

$$\begin{aligned}\sigma_{12} &= -k[c_{66}(b_1+\alpha_1)B_1 e^{-b_1 y} + c_{66}(b_2+\alpha_2)B_2 e^{-b_2 y}]\, \mathrm{e}^{ik(x_1-ct)},\\ \sigma_{22} &= ik[(c_{12} - c_{22}b_1\alpha_1)B_1 e^{-b_1 y} + (c_{12} - c_{22}b_2\alpha_2)B_2 e^{-b_2 y}]\, \mathrm{e}^{ik(x_1-ct)}.\end{aligned} \tag{27}$$

Using (27) into (3) and taking into account (4) we have

$$t_{12} = kT_1(y)e^{ik(x_1-ct)}, \quad t_{22} = kT_2(y)e^{ik(x_1-ct)}, \tag{28}$$

where

$$T_1(y) = \beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}, \quad T_2(y) = i(\gamma_1 B_1 e^{-b_1 y} + \gamma_2 B_2 e^{-b_2 y}), \tag{29}$$

with

$$\beta_j = -\frac{c_{66}(b_j+\alpha_j)}{1+e(1-b_j^2)}, \quad \gamma_j = \frac{c_{12} - c_{22}b_j\alpha_j}{1+e(1-b_j^2)}, \quad j = 1,2. \tag{30}$$

Taking $x_2 = 0$ in $(23)_2$ and (29) we obtain

$$\begin{aligned}&U_2(0) = i(\alpha_1 B_1 + \alpha_2 B_2),\\ &T_1(0) = \beta_1 B_1 + \beta_2 B_2, \quad T_2(0) = i(\gamma_1 B_1 + \gamma_2 B_2).\end{aligned} \tag{31}$$

Substituting (31) into (22) gives a homogeneous system of two linear equations for B_1 and B_2. For a non-trivial solution, the determinant of coefficients of this system must vanish. This gives the equation

$$\Big(n_{32}n_{41} - n_{31}n_{42}\Big)[\alpha;\beta] + \Big(n_{34}n_{41} - n_{31}n_{44}\Big)[\gamma;\beta] = 0. \tag{32}$$

Using (11), (24), and (30) into (32), we find

$$A_0 + B_0 \mathrm{sh}\varepsilon_1 \mathrm{sh}\varepsilon_2 + C_0 \mathrm{sh}\varepsilon_1 \mathrm{ch}\varepsilon_2 + D_0 \mathrm{ch}\varepsilon_1 \mathrm{sh}\varepsilon_2 + E_0 \mathrm{ch}\varepsilon_1 \mathrm{ch}\varepsilon_2 = 0, \tag{33}$$

where the coefficients A_0, B_0, C_0, D_0, and E_0 are determined by

$$\begin{aligned}
A_0 &= 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_1\bar{\gamma}_2(c^*_{11}-X)\Big[1+\frac{e}{c^*_{22}}(c_{12}+c_{22}+X)\Big]\sqrt{S+2\sqrt{P}},\\
B_0 &= (\bar{\beta}_1{}^2\bar{\gamma}_2^2+\bar{\beta}_2{}^2\bar{\gamma}_1^2)(c^*_{11}-X)\Big[1+\frac{e}{c^*_{22}}(c_{12}+c_{22}+X)\Big]\sqrt{S+2\sqrt{P}},\\
C_0 &= -\bar{\beta}_1\bar{\gamma}_2[\bar{\alpha};\bar{\beta}]\Big\{\big[c^2_{12}-c_{22}(c^*_{11}-X)+c_{12}Xe\big]\sqrt{P}+X(c^*_{11}-X)\frac{c_{12}e+c_{22}(1+e)}{c^*_{22}}\Big\},\\
D_0 &= \bar{\beta}_2\bar{\gamma}_1[\bar{\alpha};\bar{\beta}]\Big\{\big[c^2_{12}-c_{22}(c^*_{11}-X)+c_{12}Xe\big]\sqrt{P}+X(c^*_{11}-X)\frac{c_{12}e+c_{22}(1+e)}{c^*_{22}}\Big\},\\
E_0 &= -A_0.
\end{aligned} \tag{34}$$

Eq. (33) is the desired dispersion equation of Rayleigh waves propagating in a weakly nonlocal compressible orthotropic elastic half-space coated by a weakly nonlocal compressible orthotropic elastic layer with the sliding contact. It is clear that Eq. (33) is totally explicit. When $\varepsilon \to 0$, from (33) and (34), we have

$$\big[c^2_{12}-c_{22}(c^*_{11}-X)+c_{12}Xe\big]\sqrt{P}+X(c^*_{11}-X)\frac{c_{12}e+c_{22}(1+e)}{c^*_{22}}=0. \tag{35}$$

This equation is the dispersion equation of Rayleigh waves propagating in a weakly nonlocal compressible orthotropic half-space derived by Anh and Vinh [13]. If we neglect the nonlocality, i.e., $e=\bar{e}=0$, Eq. (33) is identical to Eq. (35) in Ref. [21], which is the secular equation of Rayleigh waves propagating in a (local) compressible orthotropic elastic half-space coated by a (local) compressible orthotropic elastic layer with the sliding contact.

In the dimensionless form, Eq. (33) takes the form

$$\bar{A}_0+\bar{B}_0\text{sh}\varepsilon_1\text{sh}\varepsilon_2+\bar{C}_0\text{sh}\varepsilon_1\text{ch}\varepsilon_2+\bar{D}_0\text{ch}\varepsilon_1\text{sh}\varepsilon_2+\bar{E}_0\text{ch}\varepsilon_1\text{ch}\varepsilon_2=0, \tag{36}$$

where

$$\begin{aligned}
\bar{A}_0 &= 2\bar{\beta}^*_1\bar{\beta}^*_2\bar{\gamma}^*_1\bar{\gamma}^*_2(e_1-ex-x)\Big[e_2+e(e_2+e_3)\Big]\sqrt{S+2\sqrt{P}},\\
\bar{B}_0 &= (\bar{\beta}^{*2}_1\bar{\gamma}^{*2}_2+\bar{\beta}^{*2}_2\bar{\gamma}^{*2}_1)(e_1-ex-x)\Big[e_2+e(e_2+e_3)\Big]\sqrt{S+2\sqrt{P}},\\
\bar{C}_0 &= -\bar{\beta}^*_1\bar{\gamma}^*_2[\bar{\alpha}^*;\bar{\beta}^*]\Big\{(e_2-ex)\big[e^2_3-e_2(e_1-ex-x)+ee_3x\big]\sqrt{P}\\
&\qquad +x(e_1-ex-x)[ee_3+(1+e)e_2]\Big\},\\
\bar{D}_0 &= \bar{\beta}^*_2\bar{\gamma}^*_1[\bar{\alpha}^*;\bar{\beta}^*]\Big\{(e_2-ex)\big[e^2_3-e_2(e_1-ex-x)+ee_3x\big]\sqrt{P}\\
&\qquad +x(e_1-ex-x)[ee_3+(1+e)e_2]\Big\},\\
\bar{E}_0 &= -\bar{A}_0,
\end{aligned} \tag{37}$$

where

$$x = \frac{\rho c^2}{c_{66}}, \quad e_1 = \frac{c_{11}}{c_{66}}, \quad e_2 = \frac{c_{22}}{c_{66}}, \quad e_3 = \frac{c_{12}}{c_{66}}, \quad \bar{e}_1 = \frac{\bar{c}_{11}}{\bar{c}_{66}}, \quad \bar{e}_2 = \frac{\bar{c}_{22}}{\bar{c}_{66}}, \quad \bar{e}_3 = \frac{\bar{c}_{12}}{\bar{c}_{66}},$$
$$r_\mu = \frac{\bar{c}_{66}}{c_{66}}, \quad r_v = \frac{c_2}{\bar{c}_2}, \quad c_2 = \sqrt{\frac{c_{66}}{\rho}}, \quad \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}, \quad e = k^2\epsilon^2, \quad \bar{e} = k^2\bar{\epsilon}^2, \tag{38}$$

and

$$\bar{\alpha}_j^* = \frac{(\bar{e}_3+1)\bar{b}_j}{1-\bar{e}r_v^2x - r_v^2x - (\bar{e}_2 - \bar{e}r_v^2x)\bar{b}_j^2}, \quad \bar{\beta}_j^* = \frac{r_\mu(\bar{b}_j - \bar{\alpha}_j^*)}{1+\bar{e}(1-\bar{b}_j^2)}, \quad \bar{\gamma}_j^* = r_\mu \frac{\bar{e}_3 + \bar{e}_2\bar{b}_j\bar{\alpha}_j^*}{1+\bar{e}(1-\bar{b}_j^2)},$$
$$S = \frac{[e_1 - x(1+e)](e_2 - ex) + [1 - x(1+e)](1-ex) - (e_3+1)^2}{(1-ex)(e_2-ex)}, \tag{39}$$
$$P = \frac{[e_1 - x(1+e)][1 - x(1+e)]}{(1-ex)(e_2-ex)},$$

with $\bar{b}_1, \bar{b}_2$ are determined by $(13)_{4,5}$ with the bar and $\bar{S}$, $\bar{P}$ are calculated by

$$\bar{S} = \frac{\left[\bar{e}_1 - r_v^2 x(1+\bar{e})\right](\bar{e}_2 - \bar{e}r_v^2x) + \left[1 - (1+\bar{e})r_v^2x\right](1-\bar{e}r_v^2x) - (\bar{e}_3+1)^2}{(1-\bar{e}r_v^2x)(\bar{e}_2 - \bar{e}r_v^2x)},$$
$$\bar{P} = \frac{\left[\bar{e}_1 - (1+\bar{e})r_v^2x\right]\left[1-(1+\bar{e})r_v^2x\right]}{(1-\bar{e}r_v^2x)(\bar{e}_2-\bar{e}r_v^2x)}. \tag{40}$$

4. NUMERICAL EXAMPLES

In order to illustrate the obtained theoretical results, some numerical examples are carried out to show the effect of the nonlocality and the thickness of layer on the Rayleigh wave velocity. The obtained numerical results are presented in Figs. 1, 2, and 3. Figs. 1 and 2 show respectively the dependence of the Rayleigh wave velocity on the dimensionless thickness of layer $\varepsilon = kh$ and the dimensionless nonlocality parameter e. Fig. 3 shows influence of the local/nonlocal combination of the layer/half-space structure on the Rayleigh wave velocity.

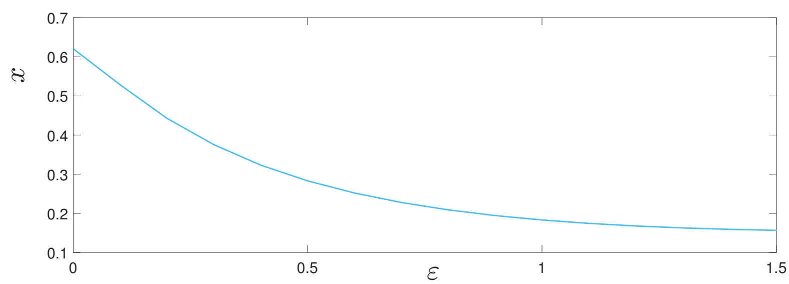

Fig. 1. Dependence of the Rayleigh wave velocity on $\varepsilon \in [0, 1.5]$. Here we take $e_1 = 2, e_2 = 2.2, e_3 = 0.5, e = 0.2$ for the half-space, $\bar{e}_1 = 2.5, \bar{e}_2 = 3.2, \bar{e}_3 = 0.8, \bar{e} = 0.3$ for the layer and $r_v = 1.8, r_\mu = 1.2$

It is seen from Fig. 1 that the Rayleigh wave velocity decreases when increasing the dimensionless thickness ε of the layer: very strongly in the interval $[0, 0.5]$, strongly in the interval $[0.5, 1]$ and then it goes slowly to the velocity of Stoneley waves propagating in two half-spaces characterized by c_{ij}, ρ, e and $\bar{c}_{ij}$, $\bar{\rho}, \bar{e}$ as $\varepsilon \to +\infty$. Fig. 2 shows that the Rayleigh wave velocity decreases when increasing the dimensionless nonlocality parameter e. This fact is agreeable to the softing behavior of strain-driven nonlocal elastic solids including weakly nonlocal elastic materials. Fig. 3 indicates that among four local/nonlocal combinations of the layer/half-space structure, the Rayleigh wave velocity is the biggest when both the layer and the half-space are local, the smallest when they are both nonlocal, the Rayleigh wave velocity for the local/nonlocal combination is smaller than the one for the nonlocal/local case. These facts say that the nonlocality decreases Rayleigh wave velocity and the nonlocality of the half-space affects the Rayleigh wave velocity more strongly than the one of the layer. It is noted that the numerical results have been verified by checking the traction-free boundary condition at $x_2 = -h$.

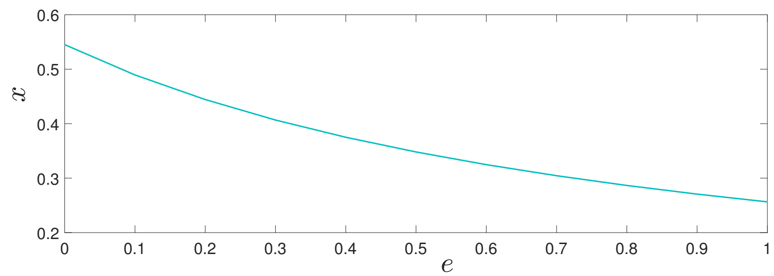

Fig. 2. Dependence of the Rayleigh wave velocity on the nonlocality $e \in [0, 1]$. Here we take $e_1 = 2, e_2 = 2.2, e_3 = 0.5$ for the half-space, $\bar{e}_1 = 2.5, \bar{e}_2 = 3.2, \bar{e}_3 = 0.8, \bar{e} = 0.1$ for the layer and $r_v = 1.8, r_\mu = 1.2$ and $\varepsilon = 0.2$

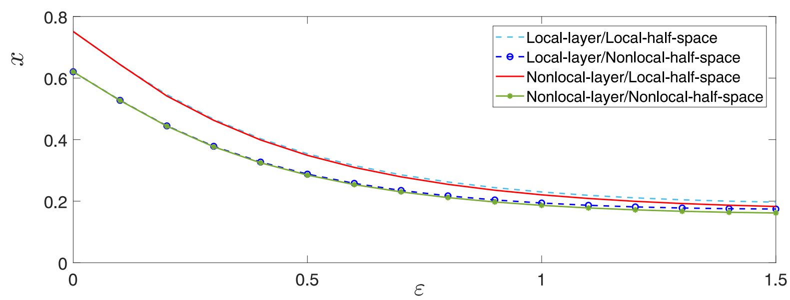

Fig. 3. Dependence of the Rayleigh wave velocity on $\varepsilon \in [0, 1.5]$ with different local/nonlocal combinations of the layer and the half-space. Here we take $e_1 = 2, e_2 = 2.2, e_3 = 0.5$, $\bar{e}_1 = 2.5, \bar{e}_2 = 3.2, \bar{e}_3 = 0.8$, $r_v = 1.8, r_\mu = 1.2$, $e = 0.2$ for the nonlocal half-space, $\bar{e} = 0.2$ for the nonlocal layer

5. CONCLUSIONS

In this paper, the propagation of Rayleigh waves in a nonlocal orthotropic half-space coated by a nonlocal orthotropic layer with sliding contact is studied. In contrast to previous studies which employed the ill-posed Eringren's nonlocal elasticity theory, this investigation used the weakly nonlocal elasticity model which has been proven to be well-posed for any harmonic plane wave problem. The dispersion equation of Rayleigh waves has been derived in the explicit form by using the transfer matrix method and the effective boundary condition method. It recovers the secular equation of Rayleigh waves propagating in a local orthotropic half-space coated by a local orthotropic layer with sliding contact. Since the dispersion equation of Rayleigh waves is totally explicit, it will be a powerful tool for monitoring the health of the layer/half-space structures during loading. Especially, it will be a convenient tool for evaluating the nonlocality parameters of layer and half-space which are difficult to determine by experiments.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

ACKNOWLEDGEMENT

The work was supported by the Vietnam National Foundation For Science and Technology Development (NAFOSTED) under Grant No. 107.02-2021.59.

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